Making the Laws of Sines and Cosines a Splash for Students

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The Laws of Sines and Cosines are tremendously powerful in solving application problems, but traditionally the use of these methods is reduced to solving static word problems out of a textbook. This article describes a way for students to apply these trigonometric methods to a very novel and motivating context of hitting their mathematics teacher with water balloon trajectories.

Introduction

In a standard trigonometry class, students are taught the Law of Sines and the Law of Cosines. For any $\triangle ABC$ with side lengths a, b, c and opposing angles A, B, C, these relationships state respectively that: $\frac{sin(A)}{a} = \frac{sin(B)}{b} = \frac{sin(C)}{c}$ and $c^2 = a^2 + b^2 - 2abcos\,(C)$.

After discovering and/or proving these statements, students then apply these two relationships to solve word problems applicable in many contexts such as architecture, surveying, and navigation. While some students find these problems interesting, designing more interactive problems can further increase student motivation and understanding.

The purpose of this article is to showcase the transformation of a traditional task to a more interactive and authentic one for students while at the same time allowing for a deeper understanding of mathematical content.

The Task

In preparing for a unit on solving triangles, I scanned a textbook for application problems, hoping to come across a problem that would either invoke student discussion or be related to another class such as Physics. I was instantly inspired by a problem involving a hunter trying to hit a moving deer. Initial distances and angles were given that created an oblique triangle solvable by the Law of Cosines. Rather than assigning this problem (which would interest only a few students who didn't find hunting inhumane), I reframed the context of the problem to make it more tangible. Rather than a moving deer, students were to be given the task to calculate when to throw a water balloon to hit their math teacher. The scenario is depicted in Figure 1.

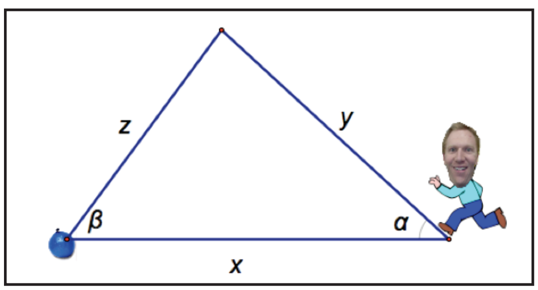

Fig 1 Diagram showing problem situation

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Day 1: Initial Measurements

This activity spanned two days. To start, students were assigned into three heterogeneous groups by ability level, each consisting of approximately three to four students. As depicted in Figure 2, each group was given a different set of initial conditions: The initial horizontal distance I was away, x, the time they had to hit me, t, and my directional angle, α. As a full class, students brainstormed what other information would be helpful for them to measure in order to be successful with the task. Students quickly determined they would need to calculate my average speed of travel as well as the average aerial speed of the water balloon.

The first day continued by taking the additional measurements stated above. Students in each group sectioned off into two subgroups. One of these subgroups determined a way to calculate my average speed. Collectively, they decided to measure the average time it took me to walk a distance of 10 meters. After 5 trials, they calculated my average speed to be approximately $v = \frac{10 \text{ meters}}{5.7 \text{ seconds}} \approx 1.75 \frac{m}{s}$. Students could then calculate my distance traveled, y, by calculating $y = v - t$, where t is the time I will travel before impact.

The difference between this calculated time and my total time t of walking solved the problem of when to throw the balloon for a direct hit!

Another subgroup of students practiced throwing water balloons past a radar gun to find a good estimate for the balloon velocity. With three trials per designated thrower, each group determined their average throwing speed, ω, most of which were approximately 30 miles per hour. The desired units for the context of the problem were in meters per second, but the radar gun only displayed speed in miles per hour. This discrepancy in units offered an authentic opportunity for students to practice unit conversions.

Day 2: Calculations and Throwing

Equipped with the first day's calculations, each group began day 2 with their own "SAS" case, specifically having the values for x, α, and y in Figure 1. Students were given approximately 20 minutes to collaborate on applying their triangle laws in context. Using the Law of Cosines, $z^2 = x^2 + y^2 - 2xycos(\alpha)$, students solved for z, the distance that the balloon would travel before impact.

Once z was determined, students could set up the Law of Sines as $\frac{sin(\beta)}{y} = \frac{sin(\alpha)}{z}$ to find the directional angle β of the trajectory of the balloon. This angle was important for the thrower to gauge where to aim for a hit. Students could use the balloon velocity ω and the calculated throwing distance z from the Law of Cosines to find a time. The difference between this calculated time and my total time t of walking solved the problem of when to throw the balloon for a direct hit!

Once calculations were completed, students ventured outside to witness the accuracy of their calculations. Two large protractors were placed on the ground separated by a horizontal distance x given to each group. The radar gun and thrower were stationed at one protractor while a tape measure emanated from the second protractor at an angle α. One student in each group had a timer and told me when to start walking. They also alerted the thrower when to release the balloon based on their prior calculations. Each group got three balloons to try to score a hit.

Conclusion

In general, students were very successful with performing the calculations, especially with such

	Group A	Group B	Group C
Horizontal Distance, x	10 meters	8 meters	12 meters
Time Traveled, t	8 seconds	7 seconds	10 seconds
Directional angle, α	65°	85°	50°

Fig 2 Table showing initial conditions per group

subtleties of unit conversion and rounding error. The context gave them a practical application of the Laws of Sines and Cosines where the accuracy of the answer had utmost importance. Moreover, students enjoyed the opportunity to try to hit their mathematics teacher on a nice day outside (see Figure 3 showing an action shot). Though only one group was successful in striking a hit, their inaccuracy allowed students to reflect on the application of a theoretical calculation to an applied context. Students noticed that the vertical component of gravity, wind resistance, and variations in both calculated speeds were other variables that could be further examined for higher accuracy. In any instance of mathematical modeling, students should draw attention to the assumptions made for the practicality of the model.

I highly encourage you to look back on application problems traditionally assigned in your class and try to redesign the problem to be a more interactive and engaging task. By framing a problem that students can model and experience, student motivation and learning increase drastically.

Fig 3 A student throwing a balloon

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